Problem set 2

November 17, 2015

Problem 1. Let $\varphi\colon \mathbb{R} \to \mathbb{R}_{>0}$ be the map give by $\varphi(x) = 2^x$. Prove that φ is a homomorphism of groups ($\mathbb{R}$ with $+$, and $\mathbb{R}_{>0}$ with $\times$).

A homomorphism is called an **isomorphism** if it is a bijection. Two groups G, H are called **isomorphic** if there is an isomorphism (maybe not unique) $\varphi\colon G \to H$.

Problem 2. A homomorphism $\varphi\colon G \to H$ of groups is an isomorphism **if and only if** there is a homomorphism $\psi\colon H \to G$ such that $\varphi \circ \psi = id_H$ and $\psi \circ \varphi = id_G$, i.e. ψ is inverse of φ.
Problem 3. Prove that $\varphi\colon \mathbb{R} \to \mathbb{R}_{>0}$ defined in the first problem is an isomorphism.
Problem 4. Are groups $\mathbb{R}$ with $+$ and $\mathbb{R}^\times = \{x \in \mathbb{R} \mid x \neq 0\}$ with $\times$ isomorphic?
Problem 5. Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

Prove that $(G \times H, *)$ is again a group. It's called the **product of groups** G, H.
Problem 6. Are the following pairs of groups isomorphic?

- $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$;
- S_3 and $\mathbb{Z}/6$;
- $\mathbb{Z}/6$ and $\mathbb{Z}/2 \times \mathbb{Z}/3$;
- $\mathbb{Z}/2 \times \mathbb{R}_{>0}$ and $\mathbb{R}^\times$.